

Notions of dependence and independence  
The basic idea of GTS

A function  $y = f(x)$  is continuous at  $x_0$  : *given* a number  $\alpha$  however small, we can find  $\epsilon$  such that  $|f(x) - f(x_0)| < \alpha$ , *given any*  $x$  such that  $|x - x_0| < \epsilon$ .

In game-theoretical terms, "we can find" becomes the existential quantifier (player), and "given any" becomes the universal quantifier (player):

$$1. \forall \alpha \exists \epsilon \forall x [(|x - x_0| < \epsilon) \rightarrow (|f(x) - f(x_0)| < \alpha)]$$

The truth of (1), i.e, the existence of a winning strategy for the existential player is given by the second-order sentence

$$2. \exists g \forall \alpha \forall x [(|x - x_0| < g(\alpha)) \rightarrow (|f(x) - f(x_0)| < \alpha)].$$

A very robust notion of truth.

## 1 Dependence and Independence of quantifiers: Branching quantifiers - *IF* logic

$$\left( \begin{array}{c} \forall x_0 \\ \exists x_1 \end{array} \right) R(x_0, x_1) \Leftrightarrow \forall x_0 (\exists x_1 / \{x_0\}) R(x_0, x_1)$$

$$\left( \begin{array}{c} \exists x_0 \\ \exists x_1 \end{array} \right) R(x_0, x_1) \Leftrightarrow \exists x_0 (\exists x_1 / \{x_0\}) R(x_0, x_1)$$

$$\left( \begin{array}{cc} \forall x_0 & \exists x_1 \\ \forall x_2 & \exists x_3 \end{array} \right) Q(x_0, \dots, x_3) \Leftrightarrow \forall x_0 \forall x_2 (\exists x_1 / \{x_2\}) (\exists x_3 / \{x_0, x_1\}) Q(x_0, \dots, x_3)$$

### Two game-theoretical paradigms

The extensive paradigm:

$$\text{Games of imperfect information: } G = (N, H, Z, P, (I_i)_{i \in N}, (u_i)_{i \in N})$$

The strategic paradigm:

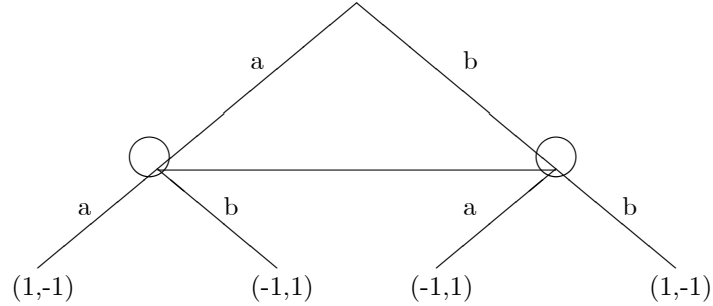
$$\text{One shot games: } G = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$$

IF-logic: The extensive paradigm

Henkin logic: The strategic paradigm

## 2 IF-logic: The extensive paradigm

The game  $\forall x_0(\exists x_1/\{x_0\})x_0 = x_1$  played on  $M = \{a, b\}$  :



A strategy is a function defined on all the earlier possible histories, uniform on the equivalent histories.

A sentence is true (in a model) if there is a uniform winning strategy for the existential player.

A sentence is false (in a model) if there is a uniform winning strategy for the universal player.

Uniformity of strategies = functional dependence on known moves = Thomson transformations

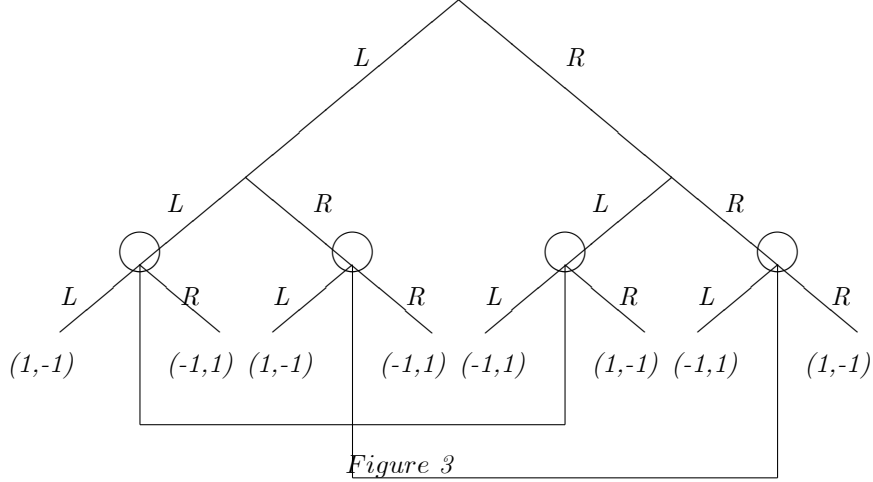
sentence is true (in a model) if there is a uniform winning strategy for the universal player.

A sentence is false (in a model) if there is a uniform winning strategy for the existential player.

$$\forall x_0(\exists x_1/\{x_0\})x_0 = x_1 \Leftrightarrow \exists f(f \text{ is } \{x_0\}\text{-uniform and } x_0 = f(x_1)) \Leftrightarrow \exists x_1 \forall x_0 x_0 = x_1$$

### 2.1 Signalling

**Example 1** *The extended password game is a simple variant of the password game with perfect information. It is played by two teams, the team of Abelard, consisting of one player,  $\forall = \{\forall_1\}$  and the team of Eloise consisting of two players,  $\exists = \{\exists_1, \exists_2\}$ .  $\forall_1$  tells  $\exists_2$  a password L or R (without player  $\exists_1$  hearing it), that  $\exists_2$  reports to  $\exists_1$ . If  $\exists_1$  is able to report the password told by  $\forall_1$ , the team  $\exists$  wins; otherwise the team  $\forall$  wins.*



The game is modelled by  $\forall x_0 \exists x_2 (\exists x_1 / \{x_0\}) x_0 = x_1$ .  
 $\forall x_0 \exists x_2 (\exists x_1 / \{x_0\}) x_0 = x_1 \Leftrightarrow \exists f \exists g (g \text{ is } \{x_0\}\text{-uniform and } \forall x_0 x_0 = g(f(x_0)))$ .  
 IF-logic  $\left\{ \begin{array}{l} \text{Games: Playability, cognitive constraints} \\ \text{Foundational issues} \end{array} \right.$

## 2.2 Playability of games

Perfect Recall: action recall + knowledge memory

Action Recall: A player never forgets his own earlier moves

Knowledge memory: if a player knows a move of his/her opponent at some stage, then he always knows it

**Example 2** (i) *Violation of action recall*

$$\exists x_0 (\exists x_1 / \{x_0\}) x_0 = x_1.$$

(ii) *Violation of knowledge memory*

$$\forall x_0 \exists x_2 (\exists x_1 / \{x_0\}) x_0 = x_1.$$

(iii) *Violation of both*

$$\forall x_0 \exists x_1 \forall x_2 (\exists x_3 / \{x_0, x_1\}) R(x_0, x_1, x_2, x_3)$$

**Theorem 3** (Sevenster, 2006) *IF-logic with perfect recall = First-order logic.*

### 2.3 Foundational issues

Combination of high non-standard representational power + non-standard inferential power

The representational power of IF-logic: it characterizes nonstandard models{

The inferential power of IF-logic: the set of contradictions is recursively enumerable

It defines its own truth-predicate

## 3 The strategic paradigm

Mixed strategies in the strategic game  $\left( \begin{array}{c} \forall x_0 \\ \exists x_1 \end{array} \right) R(x_0x_1)$ , the model  $M = \{a_1, \dots, a_n\}$  and two players  $\forall$  and  $\exists$ .

We assume a probability distribution  $\delta$  over  $M = \{a_1, \dots, a_n\}$ .

Each player  $i$  picks up an individual  $a_i$  and put it on the table. The utility of  $(a_{\exists}, b_{\forall})$  is

$$U(a_{\exists}, b_{\forall}) = \begin{cases} 1, & \text{if } (a_{\exists}, b_{\forall}) \in R^M \\ 0, & \text{if } (a_{\exists}, b_{\forall}) \notin R^M \end{cases}$$

In case there is a probability distribution  $\delta$  of over  $M$ , then the uti

## 4 Dependence and independence of terms: From IF-logic to DF-Logic

### 4.1 Dependence of terms

$$\begin{aligned} \left( \begin{array}{cc} \forall x_0 & \exists x_1 \\ \forall x_2 & \exists x_3 \end{array} \right) \varphi(x_0, \dots, x_3) &\Leftrightarrow \forall x_0 \forall x_2 \exists x_1 \exists x_3 (Dep(\{x_0\}, x_1) \wedge Dep(\{x_2\}, x_3) \wedge \varphi) \\ &\Leftrightarrow \forall x_0 \exists x_1 \forall x_2 \exists x_3 (Dep(\{x_2\}, x_3) \wedge \varphi) \\ &\Leftrightarrow \forall x_0 \exists x_1 \forall x_2 \exists x_3 (Dep(\{x_2\}, x_3) \wedge \varphi) \\ &\Leftrightarrow \forall x_2 \exists x_3 \forall x_0 \exists x_1 (Dep(\{x_0\}, x_1) \wedge \varphi) \end{aligned}$$

DF-language:

$$Rt_1, \dots, t_n \mid t_1 = t_2 \mid Dep(W, t_n) \mid \varphi \vee \psi \mid \neg\varphi \mid \exists x_n \varphi$$

DF-semantics:

We fix a model and a set  $X$  of assignments with  $dom(X) = \{x_1, \dots, x_n\}$ .

The crucial semantic clauses:

$$\begin{aligned} M \models_X^+ Dep(W, t_n) &\Leftrightarrow (\forall s, s' \in X)[s \text{ and } s' \text{ agree on } W \Rightarrow s(t_n) = s'(t_n)] \\ M \models_X^- Dep(W, t_n) &\Leftrightarrow X = \emptyset \end{aligned}$$

For sentences  $\varphi$  we define

$$M \models^+ \varphi \Leftrightarrow M \models_{\{\emptyset\}}^+ \varphi$$

$$M \models^- \varphi \Leftrightarrow M \models_{\{\emptyset\}}^- \varphi.$$

Let  $W = \{t_1, \dots, t_m\}$ . Notice that:

$$M \models_X^+ Dep(W, t_n) \iff \text{there is a function } f_{x_i} \text{ such that for all } s \in X : s(t_n) = f_{x_i}(s(t_1), \dots, s(t_m))$$

Some logically valid formulas:

$$\begin{aligned} & \forall x_0 \exists x_1 Dep(\{x_0\}, x_1) \\ & \forall x_0 \exists x_1 [Dep(\{x_0\}, x_1) \wedge x_1 = f(x_0)] \\ & \forall x_0 \forall x_1 (x_1 = f(x_0) \rightarrow Dep(\{x_0\}, x_1)) \\ & \forall x_0 \forall x_1 \forall x_2 (Dep(\{x_0\}, x_1) \wedge Dep(\{x_1\}, x_2) \rightarrow Dep(\{x_0\}, x_2)) \end{aligned}$$

## 4.2 Notions of dependence

$$x_i \text{ is dependent on } W : \begin{cases} \text{Formal dependence: } x_i \in W \\ \text{Functional dependence: Armstrong Axioms} \\ \text{Causal Dependence: Kolmogorov axioms} \\ \text{etc} \end{cases}$$

Armstrong's axioms:

1. If  $x \in W$ , then  $x$  depends on  $W$
2. If  $x$  depends on  $W$ , and each member of  $W$  depends on  $Z$ , then  $x$  depends on  $Z$ ;
3. If  $x_i$  depends on  $W$ , then there is a minimal  $U \subseteq W$  such that  $x_i$  depends on  $U$

DF-logic expresses functional dependence: Armstrong's axioms are valid.

## 4.3 Independence of terms

$$\begin{aligned} \left( \begin{array}{cc} \forall x_0 & \exists x_1 \\ \forall x_2 & \exists x_3 \end{array} \right) \varphi(x_0, \dots, x_3) & \Leftrightarrow \exists f \exists g \forall x_0 \forall x_2 \varphi(x_0, f(x_0), x_2, g(x_2)) \\ & \Leftrightarrow \forall x_0 \forall x_2 (\exists x_1 / \{x_2\}) (\exists x_3 / \{x_0, x_1\}) \varphi \\ & \Leftrightarrow \forall x_0 \forall x_2 \exists x_1 \exists x_3 (Ind(x_1, \{x_2, x_3\}) \wedge Ind(x_3, \{x_0, x_1\}) \wedge \varphi) \\ & \Leftrightarrow \forall x_0 \exists x_1 \forall x_2 \exists x_3 (Ind(x_3, \{x_0, x_1\}) \wedge \varphi) \end{aligned}$$

The same syntax and semantics as in *DF*-logic.

Two notions of independence:

Independence as functional dependence (Hintikka):

$$M \models_X^+ Ind(t_n, W) \Leftrightarrow M \models_X^+ Dep(dom(X) - (W \cup \{t_n\}), t_n)$$

**Example 4**  $dom(X) = \{x_1, x_2, x_3\}$  and  $X$  :

$x_1$	$x_2$	$x_3$
0	$a$	$r$
0	$a$	$s$
1	$c$	$r$

$x_1$  is dependent on  $\{x_2\}$ , hence  $x_1$  is independent of  $\{x_3\}$ .

Independence as lack of functional dependence:

$$M \models_X^+ Ind(t_n, W) \Leftrightarrow M \not\models_X^+ Dep(W, t_n)$$

Notice that:

$$M \models_X^+ Ind(t_n, W) \Leftrightarrow \exists s, s' \in X(s \text{ and } s' \text{ agree on } W \text{ but } s(t_n) \neq s'(t_n))$$

Notions of independence

$$x_i \text{ is independent of } W : \begin{cases} \text{Imperfect information: game-theory} \\ \text{Algebraic independence} \\ \text{Randomness} \end{cases}$$

Imperfect information-Thomson's transformations

The following is a valid *IF*-formula:

$$\forall x \exists y Ind(y, W) \Leftrightarrow \exists y \forall x Ind((y, W - \{x\}))$$

#### 4.4 Independence as randomness (Van Lambalgen's *IF*-logic)

A relation of infinitary arity:

$$R(x, W) : x \text{ is independent of } W$$

1. Axioms and rules for inference for classical predicate logic
2.  $\exists x R(x, \emptyset)$
3.  $R(x, \vec{y} \vec{z}) \rightarrow R(x, \vec{z})$
- 4a.  $R(x, \vec{y}) \rightarrow R(x, \pi \vec{y})$  for any permutation  $\pi$ ;
- 4b.  $R(x, y, \vec{z}) \rightarrow R(x, yy \vec{z})$
5.  $R(x, y) \rightarrow x \neq y$
6.  $\exists x R(x, \vec{y}) \wedge \varphi(x, \vec{y}) \rightarrow \exists x (R(x, z \vec{y}) \wedge \varphi(x, \vec{y}))$
7.  $R(y, \vec{z}) \wedge R(x, y \vec{z}) \rightarrow R(y, x \vec{z})$  (Steinitz exchange principle)

In *IF*-logic, 4a and 4b come for free. Also  $Ind(x_n, W)$  satisfies all the other axioms, with the exception of (6):

$$\begin{aligned} & \models \exists x Ind(x_i, \emptyset) \\ & \models Ind(x_i, W \cup Z) \rightarrow Ind(x_i, Z) \\ & \models Ind(x, \{y\}) \rightarrow x \neq y \\ & \models Ind(y, Z) \wedge Ind(x, \{y\} \cup Z) \rightarrow Ind(y, \{x\} \cup Z) \end{aligned}$$

In particular, we draw attention to Steinitz exchange principle. One of its instances is:

$$8. \text{Ind}(x_1, \{x_2\}) \wedge \text{Ind}(x_3, \{x_1, x_2\}) \rightarrow \text{Ind}(x_1, \{x_2, x_3\})$$

Fix an  $X$ , say  $\text{dom}(X) = \{x_1, \dots, x_5\}$ .

Then (8) becomes:

$$9. \text{Dep}(\{x_3, x_4, x_5\}, x_1) \wedge \text{Dep}(\{x_4, x_5\}, x_3) \rightarrow \text{Dep}(\{x_4, x_5\}, x_1)$$

(9) is a valid principle in  $DF$ -logic.

Axiom (6) requires another notion of independence:  $x_n$  is not functionally dependent on  $W$ . That is, a consequence of (6) is:

$$\forall x(R(x, \vec{y}) \rightarrow x \neq f(\vec{y})) \text{ for every definable } f.$$

A counter-example to (6) in  $DF$ -logic.

Take  $\text{dom}(X) = \{x_1, x_2, x_3\}$  and  $X$  :

$x_1$	$x_2$	$x_3$
$a$	$a$	$a$
$a$	$b$	$b$
$b$	$c$	$c$

**Example 5** Let  $f$  be a function such that  $f(a) = f(b) = a, f(c) = b$ . We have

$$M \models_X^+ \text{Ind}(x_1, \{x_3\}) \wedge x_1 = f(x_3).$$

But on the other side the following reasoning takes place:

Assume  $\exists x_1 \text{Ind}(x_1, \{x_3\}) \wedge x_1 = f(x_3)$ . Put  $z = f(x_3)$ . Then from (6) we get  $\exists x_1 \text{Ind}(x_1, \{z, x_3\}) \wedge x_1 = f(x_3)$  which contradicts (5).

The independence of  $x_1$  from  $\{x_2, x_3\}$  would have required that all the assignments agree on  $x_1$ .

(Thomson's Transformations)

Notice an immediate property of transitivity:

Assume

1.  $x_i$  is dependent on  $W, x_i \notin W$
2.  $x_j$  is independent from  $W$
3.  $x_i \neq x_j$
4.  $\text{dom}(X) = \{x_i, x_j\} \cup W$

Then  $x_i$  is dependent on  $W$

Note that from our definitions it follows that  $x_j \notin W$ . The proof is a direct corollary of transitivity of dependence: the second assumption becomes:  $x_j$  is dependent on  $x_i$  which is dependent on  $W$ , hence  $x_j$  is dependent on  $W$ .

Here is a picture:

$x_1$	$x_2$	$x_3$
0	$a$	$r$
1	$b$	$s$
2	$c$	$r$

$x_2$  dependent of  $\{x_1\}$   
 $x_3$  independent of  $\{x_1\}$   
hence:  $x_3$  dependent of  $\{x_1\}$

## 5 Definability of truth

### 5.1 Tarski's negative results

For a language  $L$  which defines its own syntax:

(a) it is impossible for  $L$  to contain an adequate truth predicate for  $L$  such that the  $T$ -equivalences are logically deducible from the principles of elementary syntax of  $L$ ;

(b) it is not even possible for  $L$  to contain a truth predicate for  $L$  whose extension would coincide with the set of sentences of  $L$  that are true in an interpretation  $M$ .

Kripke's result: a solution to (b)

Hintikka and Sandu: a solution to (a)

Philippe de Rouilhan and Serge Bozon: Two major stages in the history of post-Tarskian truth theories.

Kripke 1975 brings a partial solution to (b):

A language  $L$  and a model  $M$  so that

- $L$  defines its own syntax
- $L$  contains a truth-predicate  $Tr$
- The extension of  $Tr$  in  $M$  coincides with the set of sentences true in  $M$  (e-adequacy)
- Thus Kripke's truth-predicate is e-adequate.

Hintikka & Sandu (1996) bring a partial solution to (a):

A language  $L$  so that:

- $L$  defines its own syntax
- $L$  contains a truth-predicate  $Tr$

- The extension of  $Tr$  in every model of  $PA$  coincides with the set of sentences true in  $M$ .
- Thus Hintikka and Sandu's truth-predicate is  $i$ -adequate.

But (de Rouilhan, Bozon, Feferman):

- The  $i$ -adequacy can be shown in the second-order metalanguage:
- There is no calculus for  $IF$ -logic: the class of logical consequences of an  $IF$ -theory are not recursively enumerable.

Thus:

- the truth-predicate cannot be used by somebody whose working language is  $IF$ .

Answer:

- every  $IF$ -theory (which contains  $PA$ ) can be extended to a theory which defines its own logical consequences.
- $IF$  logic has a proof system for contradictions:

$$\begin{aligned} &\text{First-order logic:} \\ &\Gamma \models \varphi \iff \Gamma \vdash \varphi \\ &\models \varphi \iff \vdash \varphi \end{aligned}$$

$$\begin{aligned} &\text{IF-logic logic:} \\ &\Gamma \cup \{\varphi\} \models \perp \iff \Gamma \cup \{\varphi\} \vdash_{IF} \perp \\ &\varphi \models \perp \iff \varphi \vdash_{IF} \perp \end{aligned}$$

- In first-order language (enriched with a truth-predicate) one can express things like

$$\forall x (Pr_{\Gamma}(x) \rightarrow Tr(x))$$

- In IF-logic which defines its own truth-predicate, one can express things like

$$\forall x (Pr_{\Gamma}^{IF}(x) \rightarrow \neg Tr(x))$$